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The growth of Görtler vortices in compressible boundary layers

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Abstract. The linear instability of Görtler vortices in compressible boundary layers is considered. Using asymptotic methods in the high-wavenumber regime, it is shown that a growth-rate estimate can be found by solving a sequence of linear equations. The growth rate obtained in this way takes non-parallel effects into account and can be found much more easily than by ordinary differential equation eigenvalue calculations associated with parallel-flow theories.

1. Introduction

Our concern is with the linear growth of Taylor–Görtler vortices in compressible boundary layers. We develop a simple method for generating curves of constant amplification rates in the high-wavenumber regime.

The growth of Taylor–Görtler vortices in incompressible boundary layers has received a lot of attention in recent years due to its relevance to Laminar Flow Control (see for example Harvey and Pride [1]). The original calculation by Görtler [2] showed that Taylor's [3] instability mechanism which occurs for curved flows also operates in boundary-layer flows. However, the relative complexity of the basic state for a boundary-layer flow makes it a much more difficult task to examine the instability of this state theoretically. Thus, the essential difficulty with the linear instability problem is that the growth of the boundary layer cannot in general be ignored and the appropriate linear instability equations are therefore partial differential equations.

The original calculation by Görtler ignored the effect of boundary-layer growth completely and his numerical results were later corrected by Hammerlin [4] who found that instability occurs first at zero wavenumber. Later calculations by Hammerlin [5] and Smith [6] attempted to remedy this deficiency by including higher-order curvature terms or terms associated with the nonparallel nature of the basic state. Further work by Herbert [7] for example was aimed at understanding why the various linear theories did not give consistent results.

Floryan and Saric [8] gave a multiple-scale approach to the linear Görtler instability problem along the lines of, for example, the work of Gaster [9] or Saric and Nayfeh [10] for Tollmien–Schlichting waves. Thus, Floryan and Saric derived the partial differential equations governing the growth of Görtler vortices. These equations had been given in a more general context some years earlier by Gregory, Stuart and Walker [11] who discussed the instability of three-dimensional boundary layers. The equations can also be inferred from, the work of Smith [6]. The solution given by Floryan and Saric [8] followed the approach of previous investigations and implicity made a parallel-flow approximation. By the latter phrase we mean that some intrinsic property of the nonparallel nature of the basic state was ignored in solving the disturbance equations. In fact, the above authors replace streamwise partial derivatives of the vortex by local spatial growth rates thus reducing the system to a set of ordinary differential equations. It is not clear how such an approach can be justified but when the growth rate vanishes the solution can be interpreted as a local Taylor-series solution of the full partial differential equations. The relevance of the solution elsewhere is not immediately apparent.

More recently, Hall [12, 13] has shown how asymptotic and numerical methods can be used to take non-parallel effects into account in a self-consistent manner. In the first paper, it was shown that small-wavelength Görtler vortices at high Görtler numbers can be described asymptotically using a multiple-scale method. Hall found that the vortices locate themselves so as to maximize their local spatial growth rate. This requires that the vortices are concentrated in a viscous layer in the interior of the flow. The first investigation of the large-wavenumber limit approximation in the Taylor vortex problem was given by Meksyn [14].

In the linear regime, Hall [13] solved numerically the full partial differential instability equations at O(1) wavenumbers. The linear equations were found to be parabolic in the streamwise direction so that an initial disturbance was imposed at some location and its development followed as the equations were marched downstream. The growth of the disturbance was followed by calculating the local rate of change of a disturbance energy density. The neutral position was defined to be the location where this local growth rate vanished. Not surprisingly, it was found that this position was a function of the location and form of the initial disturbance. Thus it was concluded that there exists no unique neutral curve for the Görtler problem. However, at high wavenumbers, the different neutral curves merge into the asymptotic and parallel-flow neutral curves. The same would be true for the different possible growth-rate curves.

Thus, in the only regime where analytical progress is possible, the growth rate can be written down in asymptotic form and no numerical eigenvalue calculations are required. It is this idea which we will now apply to compressible boundary layers to show how growth rates for these flows can be simply calculated.

Previous calculations of the compressible Görtler problem have used the parallel-flow assumption to reduce the instability problem to an eigenvalue problem associated with an eigth-order differential system. (See for example Aihara [15], Kobayashi and Kohama [16] or El-Hady and Verma [17].) In particular, El-Hady and Verma formulated the linear stability problem along the lines of Floryan and Saric [8] and gave curves of constant growth rate for various flow conditions. We show how these curves can be generated much more simply in the only regime where they are meaningful. The method we use is based on the asymptotic theory of Hall [12] for the incompressible problem. The method can be easily used for any flow configuration and needs little computational power. The method is based on the assumption that the vortex wavelength is small compared to the boundary-layer thickness. The range of validity of the methods can only be checked by a numerical solution of the full partial differential system governing the growth of vortices in growing compressible boundary layers. However, in general the high-wavenumber regime is ultimately always applicable to any constant-wavelength disturbance vortex developing in a growing boundary layer and so is therefore always physically relevant.

The procedure adopted in the rest of this paper is as follows: in Section 2, we formulate the partial differential system governing small Görtler vortex disturbances in compressible boundary layers. In Section 3, we solve these equations for large wavenumbers and determine the spatial growth rates of the disturbances. Finally, in Section 4 we present our results and draw some conclusions.

2. Formulation of the instability equations

Apart from some minor differences, our formulation is essentially the same as that of El-Hady and Verma [17] and so the reader is referred to that paper for more details. We choose L to be a typical streamwise length scale and take ν_∞, U_∞, ϱ_∞, T_∞, μ_∞ to be the scales for the kinematic viscosity, velocity, density, temperature, and coefficient of viscosity respectively. If the curvature of the wall at the streamwise location x^* is $A^{-1}\kappa(x^*/L)$, we define the curvature parameter δ by

$$\delta = \frac{L}{A}, \tag{2.1}$$

and a Reynolds number R by

$$R = \frac{U_\infty L}{\nu_\infty}, \tag{2.2}$$

and consider the limit $R \to \infty$ with the Görtler number

$$G = 2R^{1/2}\delta \tag{2.3}$$

held fixed. The free-stream Mach number M_∞ is defined by

$$M_\infty = \frac{U_\infty\sqrt{c_v}}{\sqrt{c_p \mathscr{R} T_\infty}} \tag{2.4}$$

where c_p, c_v and $\mathscr{R}$ are the specific heats and gas constant respectively. We define (x, y, z) to be dimensionless variables in the streamwise, normal and spanwise directions scaled on L, $R^{-1/2}L$, and $R^{-1/2}L$, respectively. We shall assume that the vortices grow spatially in the x direction and therefore we consider them to be steady.

Now we consider the limit $R \to \infty$ with G and M_∞ held fixed. The fluid is assumed to satisfy the ideal-gas law whilst viscosity is taken to be a function of temperature. In a layer of depth $R^{-1/2}$ the basic state is

$$(u, v, w) = U_\infty(\bar{u}(x, y), \quad R^{-1/2}\bar{v}(x, y), 0) + \ldots,$$

$$T = T_\infty \bar{T}(x, y), \quad \mu = \mu_\infty \bar{\mu}(x, y),$$

$$\varrho = \varrho_\infty \bar{\varrho}(x, y), \quad p = \varrho_\infty U_\infty^2 \bar{p}(x) + \ldots$$

where

$$\bar{\varrho}\{\bar{u}\bar{u}_x + \bar{v}\bar{u}_y\} = (\bar{\mu}\bar{u}_y)_y - \bar{p}_x, \tag{2.5a}$$

$$(\bar{\varrho}\bar{u})_x + (\bar{\varrho}\bar{v})_y = 0, \tag{2.5b}$$

$$\bar{\varrho}[\bar{u}\bar{T}_x + \bar{v}\bar{T}_y] = \frac{1}{\Gamma}\frac{\partial}{\partial y}(\bar{\mu}\bar{T}_y)_y + (\gamma - 1)M_\infty^2\bar{\mu}\bar{u}_y^2, \tag{2.5c}$$

$$\bar{T} = T_w, \quad \bar{u} = \bar{v} = 0 \quad \text{at} \quad y = 0, \tag{2.5d}$$

together with conditions on $\bar{T}$ and $\bar{u}$ as $\bar{y} \to \infty$. Here Γ is the Prandtl number whilst T_w is the wall temperature.

We now perturb (2.5) to a disturbance periodic in the z direction. The velocity components of the disturbance and temperature are scaled in an identical manner to the corresponding basic state quantities. The linearized instability equations obtained by neglecting terms of relative order $R^{-1/2}$ are formed to be

$$\frac{\bar{p}}{\bar{T}}(\bar{u}U)_x + \bar{\mu}a^2U + \frac{\bar{p}}{\bar{T}}\bar{v}U_y - (\bar{\mu}U_y)_y + \frac{\bar{p}}{\bar{T}}\bar{u}_yV$$
$$- \left[\frac{\bar{p}}{\bar{T}^2}(\bar{u}\bar{u}_x + \bar{v}\bar{u}_y) + (\tilde{\mu}\bar{u}_y)_y\right]T - \tilde{\mu}\bar{u}_yT_y = 0, \tag{2.6a}$$

$$\frac{\bar{p}}{\bar{T}}(\bar{v}_x + \kappa\bar{u}G)U - c\bar{\mu}_yU_x - (c + 1)\bar{\mu}U_{xy} - \bar{\mu}_xU_y$$
$$+ \frac{\bar{p}}{\bar{T}}(\bar{v}V)_y + \bar{P}\frac{\bar{u}}{\bar{T}}V_x + \bar{\mu}a^2V - (c + 2)(\bar{\mu}V_y)_y + P_y$$
$$- \left[\frac{\bar{p}}{\bar{T}^2}(\bar{u}\bar{v}_X + \bar{v}\bar{v}_y + 1/2\kappa G\bar{u}^2) + (c + 1)\tilde{\mu}\bar{u}_{xy} + c\tilde{\mu}_y\bar{u}_x + (c + 2)(\tilde{\mu}\bar{v}_y)_y + \tilde{\mu}_x\bar{u}_y\right]$$
$$\times\, T - \tilde{\mu}\bar{u}_yT_x - [c\tilde{\mu}\bar{u}_x + (c + 2)\tilde{\mu}\bar{v}_y]T_y - c\bar{\mu}_y\mathrm{i}aW - (c + 1)\mathrm{i}a\bar{\mu}W_y = 0, \tag{2.6b}$$

$$\bar{\mu}_x\mathrm{i}aU + (c + 1)\bar{\mu}\mathrm{i}aU_x + \bar{\mu}_y\mathrm{i}aV + (c + 1)\bar{\mu}\mathrm{i}aV_y - \mathrm{i}aP + c\tilde{\mu}(\bar{u}_x + \bar{v}_y)\mathrm{i}aT$$
$$- \frac{\bar{u}}{\bar{T}}\bar{p}W_x - (c + 2)\mu a^2W - \frac{\bar{v}}{\bar{T}}\bar{p}W_y + (\bar{\mu}W_y)_y = 0, \tag{2.6c}$$

$$\left(\frac{\bar{p}U}{\bar{T}}\right)_x + \left(\frac{\bar{p}V}{\bar{T}}\right)_y + \mathrm{i}a\left(\frac{\bar{p}W}{\bar{T}}\right) - \left(\frac{\bar{p}T\bar{u}}{\bar{T}^2}\right)_x - \left(\frac{\bar{p}T\bar{V}}{\bar{T}^2}\right)_y = 0, \tag{2.6d}$$

$$\frac{\bar{P}}{\bar{T}}\bar{T}_x U - 2(\gamma - 1)M_\infty^2 \bar{\mu}\bar{u}_y U_y + \frac{\bar{P}}{\bar{T}}\bar{T}_y V$$

$$- \left[\frac{\bar{P}}{\bar{T}^2}(\bar{u}\bar{T}_x + \bar{v}\bar{T}_y) + (\gamma - 1)M_\infty^2 \tilde{\mu}\bar{u}_y^2 + \frac{1}{\Gamma}(\tilde{\mu}\bar{T}_y)_y\right] T$$

$$+ \frac{\bar{P}}{\bar{T}}\bar{u}T_x + \frac{\bar{\mu}}{\Gamma}a^2 T + \left(\frac{\bar{P}}{\bar{T}}\bar{v} - \frac{1}{\Gamma}\tilde{\mu}\bar{T}_y\right) T_y - \frac{1}{\Gamma}(\bar{\mu}T_y)_y = 0. \qquad (2.6e)$$

Here U, V, W, P, T denote disturbed velocity components, pressure, and temperature whilst $\tilde{\mu} = \mathrm{d}\bar{\mu}/\mathrm{d}\bar{T}$ and a is the spanwise wavenumber. We note that the above equations can be simplified if $\bar{p}$ is independent of x; in that case, we can set $\bar{p} = 1$ in which case (2.6) reduce to the equations of El-Hady and Verma [17]. The Görtler number G is defined by (2.3), $c = \bar{\lambda}/\bar{\mu}$ where $\bar{\lambda}$ is the bulk viscosity. Equations (2.6) are to be solved subject to the perturbation quantities vanishing at $y = 0, \infty$. The disturbance equations (2.6 a–e) depend on κ through the centrifugal terms proportonal to G, for simplicity we now restrict our attention to the case of constant curvature so that $\kappa = 1$.

3. The high-wavenumber solution for $M_\infty \sim O(1)$

It is known from the work of Hall [12, 13] that small-wavelength Görtler vortices are located in the boundary layer so as to maximize their local spatial amplification rate. For the incompressible case and zero amplification rate, this position corresponds to where Rayleigh's criterion is most violated. The depth of this layer is $O(a^{-1/2})$ so we define η by

$$\eta = \{y - \bar{y}(x)\}a^{1/2}, \qquad (3.1)$$

where $\bar{y}(x)$ is the as yet undetermined location of the layer. The stream-wise disturbance velocity in this layer expands as

$$U = [U_0(\eta, x) + a^{-1/2}U_1(\eta, x) + \ldots]\exp\left\{a^2\int^x \sigma(x)\,\mathrm{d}x\right\}$$

where

$$\sigma(x) = \beta_0(x) + a^{-1}\beta_1(x) + \ldots.$$

Similar expansions hold for Va^{-2}, $Wa^{-3/2}$, $Pa^{-5/2}$, and T whilst $\bar{u}$ expands as

$$\bar{u} = \bar{u}_0(x) + \bar{u}_1\eta a^{-1/2} + \ldots.$$

Again, similar expansions for $\bar{T}$, $\bar{v}$ and $\bar{\mu}$ hold. The details of the expansion procedure are essentially identical to those of Hall [12] so we shall omit a lot of detail here. The Görtler number G expands as

$$G = g_0 a^4. \qquad (3.2)$$

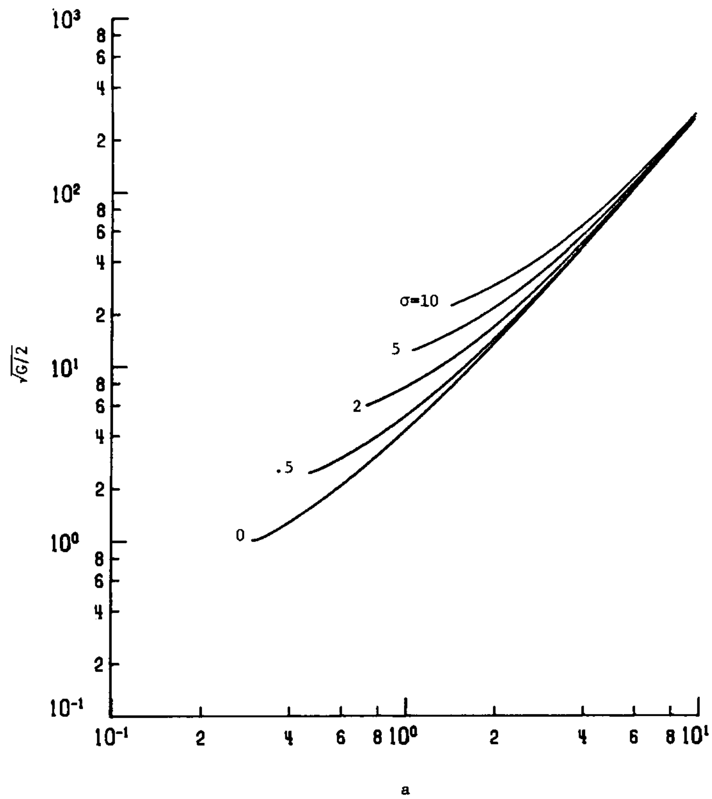

Fig. 1. Curves of constant growth σ in Görtler number *vs.* wavenumber plane for $M = 2$.

It is convenient to define the matrix A by

$$A(x, y) = \begin{bmatrix} \bar{\mu}_0 + \dfrac{\bar{u}_0 \beta_0}{\bar{T}_0} & \dfrac{\bar{u}_1}{\bar{T}_0} & 0 \\[2ex] \dfrac{g_0 \bar{u}_0}{\bar{T}_0} & \bar{\mu}_0 + \dfrac{\bar{u}_0 \beta_0}{\bar{T}_0} & \dfrac{-g_0 \bar{u}_0^2}{2\bar{T}_0^2} \\[2ex] 0 & \dfrac{\bar{T}_1}{\bar{T}_0} & \dfrac{\bar{\mu}_0}{\Gamma} + \dfrac{\bar{u}_0 \beta_0}{\bar{T}_0} \end{bmatrix}. \tag{3.3}$$

In equations (3.3) and subsequently, $\bar{p}$ has been set to unity since, for the asymptotic solution, pressure may be rescaled by local edge pressure. If the above expressions are substituted into the disturbance equations and like powers are equated, we obtain the system of equations

$$A(x, \bar{y}) \begin{Bmatrix} U_0 \\ V_0 \\ T_0 \end{Bmatrix} = 0, \tag{3.4}$$

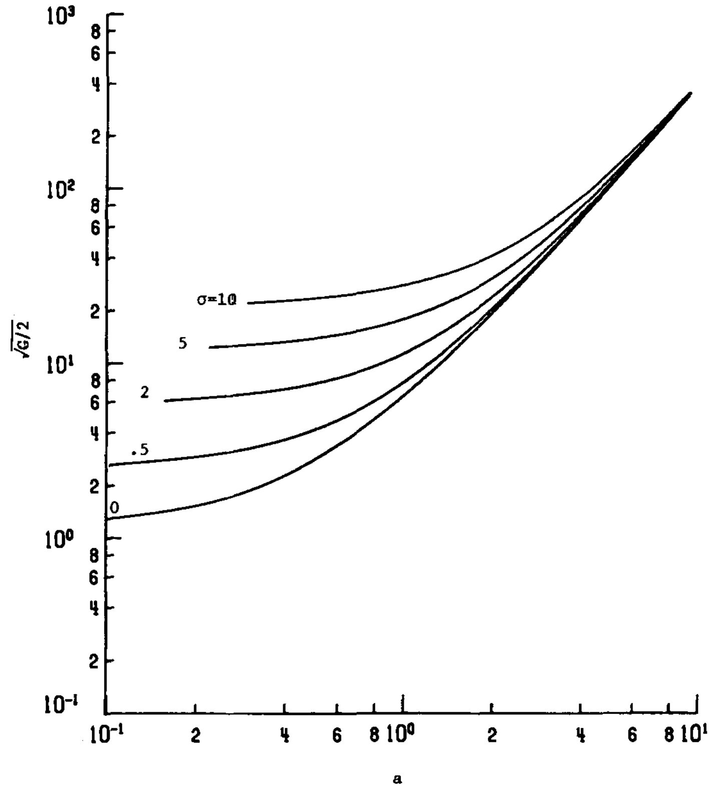

Fig. 2. Same as Fig. 1 except for $M = 4$.

at zeroth order. The functions W_0 and P_0 can be expressed in terms of U_0, V_0. The next two order systems yield

$$A(x, \bar{y}) \begin{Bmatrix} U_1 \\ V_1 \\ T_1 \end{Bmatrix} = -\eta \frac{\partial A}{\partial y}(x, \bar{y}) \begin{Bmatrix} U_0 \\ V_0 \\ T_0 \end{Bmatrix} \tag{3.5a}$$

and

$$A(x, \bar{y}) \begin{Bmatrix} U_2 \\ V_2 \\ T_2 \end{Bmatrix} = \eta^2 C(x, \bar{y}) + E \begin{Bmatrix} U_{0\eta\eta} \\ V_{0\eta\eta} \\ W_{0\eta\eta} \end{Bmatrix} + \beta_1 F \begin{Bmatrix} U_0 \\ V_0 \\ T_0 \end{Bmatrix}. \tag{3.5b}$$

The coefficient matrices C, E, and F can be written down in terms of quantities involving basic flow quantities. The system (3.4) has a nontrivial solution if

$$|A| = 0, \tag{3.6}$$

which for a given choice of g_0 and $\bar{y}$ determines three possible spatial amplification rates β_0.

The first-order solution can then be written as

$$\begin{Bmatrix} U_0 \\ V_0 \\ T_0 \end{Bmatrix} = U_0(\eta, x) \begin{Bmatrix} 1 \\ \dfrac{-\bar{T}_0}{\bar{u}_1}\left(\dfrac{\bar{u}_0\beta_0}{\bar{T}_0} + \bar{\mu}_0\right) \\ \dfrac{\bar{T}_1\left(\dfrac{\bar{u}_0\beta_0}{\bar{T}_0} + \bar{\mu}_0\right)}{\bar{u}_1\left(\dfrac{\bar{u}_0\beta_0}{\bar{T}} + \dfrac{\bar{\mu}_0}{\Gamma}\right)} \end{Bmatrix} = U_0\mathbf{a}. \tag{3.7}$$

The system

$$A^{TR}\mathbf{u}_0^+ = 0$$

will then have a solution and will be needed at higher order. Here,

$$u_0^+ = \begin{Bmatrix} 1 \\ -\left(\dfrac{\bar{u}_0\beta_0}{\bar{T}_0} + \bar{\mu}_0\right)\dfrac{\bar{T}}{\bar{u}_0 g_0} \\ \dfrac{-\bar{u}_0\left(\dfrac{u_0\beta_0}{\bar{T}_0} + \bar{\mu}_0\right)}{2\bar{T}_0\left(\dfrac{\bar{u}_0\beta_0}{\bar{T}_0} + \dfrac{\bar{\mu}_0}{\Gamma}\right)} \end{Bmatrix}$$

At this stage $\bar{y}(x)$ and $U_0(\eta, x)$ remain undetermined but at next order we find that (3.5) has a solution only if

$$(u_0^+)^{TR}\frac{\partial A}{\partial y}(x, \bar{y})\mathbf{a} = 0 \tag{3.8}$$

and this fixes the location $\bar{y}(x)$. Physically (3.8) can be interpreted as the condition that β_0 has a maximum at the layer $y = \bar{y}$. The solution of (3.8) can then be written in the form

$$\begin{Bmatrix} U_1 \\ V_1 \\ T_1 \end{Bmatrix} = \eta U_0 \begin{Bmatrix} 0 \\ \alpha \\ \beta \end{Bmatrix}$$

where (α, β) satisfies

$$\begin{aligned} \frac{\bar{T}_1\alpha}{\bar{T}_0} + \left(\frac{\bar{\mu}_0}{\Gamma} + \frac{\bar{u}_0\beta_0}{\bar{T}_0}\right)\beta &= \left(0, \frac{\bar{T}_1^2}{\bar{T}_0^2} - \frac{2\bar{T}_2}{\bar{T}_0}, \frac{\beta_0}{\bar{T}_0}\left(\frac{\bar{u}_0}{\bar{T}_0}\bar{T}_1 - \bar{u}_1\right) - \frac{\bar{\mu}_1}{\Gamma}\right)\cdot\mathbf{a}, \\ \frac{\bar{u}_1\alpha}{\bar{T}_0} &= \left(-\frac{\beta_0}{\bar{T}_0}\left(\bar{u}_1 - \frac{\bar{u}_0\bar{T}_1}{\bar{T}_0}\right) - \bar{\mu}_1, \left(\frac{\bar{u}_1\bar{T}_1}{\bar{T}_0} - 2\bar{u}_2\right)\Big/\bar{T}_0, 0\right)\cdot\mathbf{a}. \end{aligned} \tag{3.9}$$

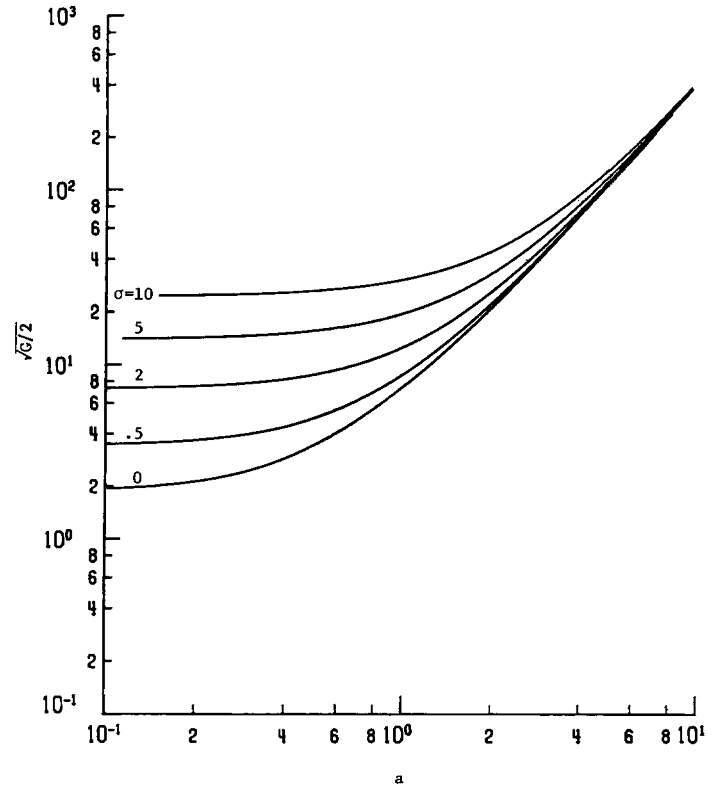

Fig. 3. Same as Fig. 1 except for $M = 6$.

Here

$$\bar{u}_n = \frac{\bar{u}^{(n)}(\bar{y})}{n!}, \quad \bar{T}_n = \frac{\bar{T}^{(n)}(\bar{y})}{n!}, \text{ etc.}$$

Finally, U_0 is determined when the required solvability condition is applied to (3.6); this yields

$$\varepsilon U_{0\eta\eta} + \gamma\eta^2 U_0 + \lambda\beta_1 U_0 = 0. \tag{3.10}$$

Here ε, γ, and λ are defined by

$$\varepsilon = (\mathbf{u}_0^+)^{TR} E\mathbf{a}, \quad \gamma = (\mathbf{u}_0^+)^{TR} C\mathbf{a}, \quad \lambda = (\mathbf{u}_0^+)^{TR} F\mathbf{a} \tag{3.11a,b,c}$$

and depend on x. The coefficient matrices E, C and F are given in Appendix A. The solutions of (3.10) which decay to zero when $|\eta| \to \infty$ can be written down in terms of parabolic cylinder functions, the most unstable one is

$$U_0 = \exp\left\{-\frac{\eta^2}{2}\sqrt{\frac{-\gamma}{\varepsilon}}\right\} \tag{3.12}$$

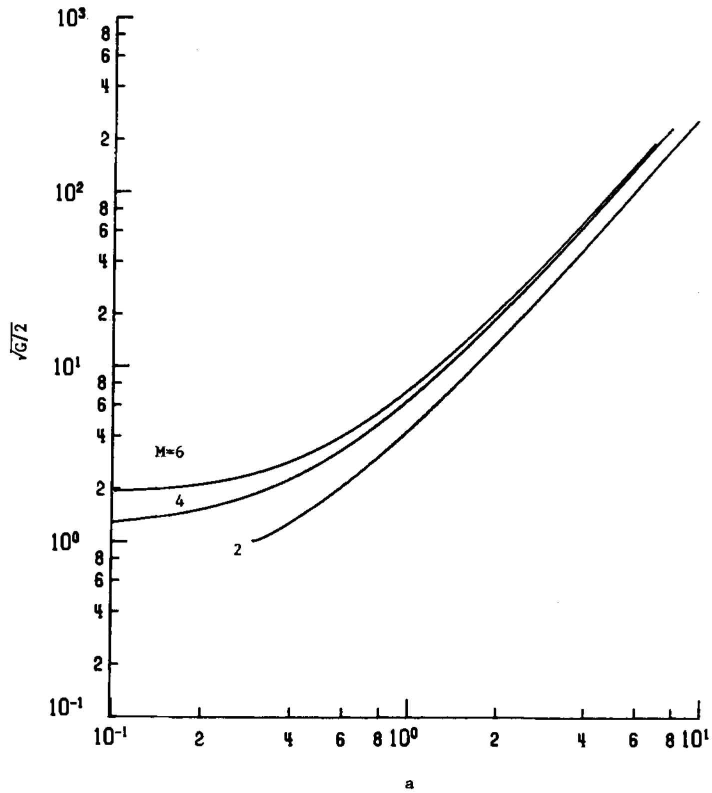

Fig. 4. Effect of compressibility on neutral ($\sigma = 0$) curves.

and the corresponding eigenrelation is

$$\beta_1 = \sqrt{-\gamma\varepsilon}/\lambda. \tag{3.13}$$

If we were interested in finding the neutral Görtler number, we would have included higher-order terms in (3.2) and set $\beta_0 = \beta_1 = 0$. In that case, (3.13) would be replaced by an equation to find g_1, the order a^3 term in the expansion of the neutral Görtler number.

We now summarize the steps required to find β_0 and β_1 – the first two terms in the expansion of the spatial amplification rate. Firstly, at any depth $\bar{y}$, the cubic equation specified by the condition

$$|A| = 0$$

is solved for the three possible values of β_0. The value of $\bar{y}$ is then varied until (3.8) is satisfied and then β_1 is determined by (3.13). Thus, it is not necessary to solve any differential equations numerically to obtain β_0 and β_1. The answer we obtain is formally valid when $a \gg 1$, at smaller values of a it is at least as valid as the solution of El-Hady and Verma [17] which would require large amounts of computer time. At progressively higher values of a, the different approaches will converge. At finite values of the wavenumber a full numerical solution of (2.6) along the lines of Hall [13] is required. We now turn to the results we have obtained using the above approach.

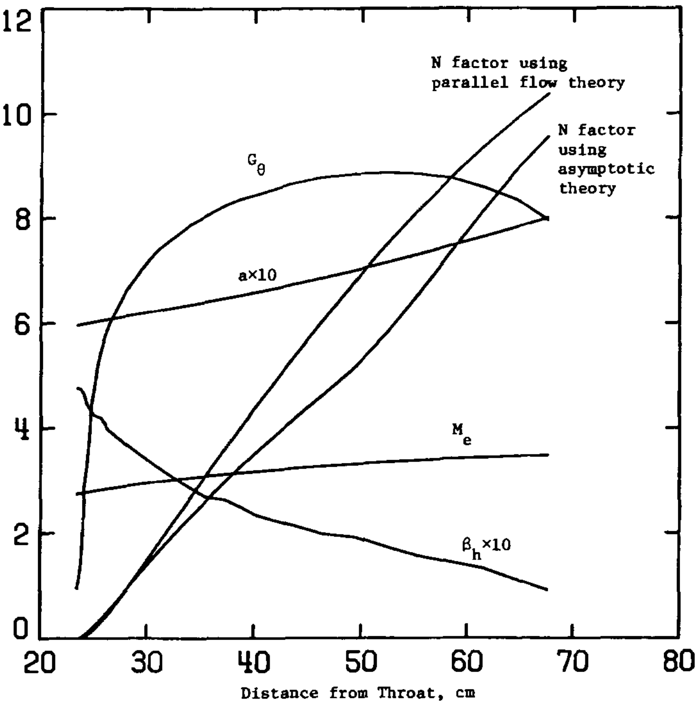

Fig. 5. Variation of local Mach number (M_e), Hartree pressure gradient parameter (β_h), Görtler number based on momentum thickness (G_θ), wave number (a) and N factors in a Mach 3.5 supersonic nozzle.

4. Results and discussion

In Fig. 1, we have shown curves of equal spatial amplification rate at a Mach number of 2 in the wavenumber *vs*. Görtler-number plane. These curves correspond to the adiabatic wall condition being applied to the temperature. The stagnation temperature is held at 311 K. The results shown are formally valid at large values of a and are then the only unique amplification rates which exist for the Görtler problem. In this figure the wavenumber is made non-dimensional by using $L = \sqrt{v_e x^*/u_e}$. Similar curves for Mach number of 4 and 6 are given in Figs 2 and 3 respectively. We note that the constant-growth-rate curves extend to small wavenumbers for higher Mach numbers. However, the small-wavenumber region is perhaps beyond the range of validity of the theory. The neutral curves for all three Mach numbers are shown in Fig. 4. These curves shift towards the left with increasing Mach number, indicating the stabilizing effect of compressibility.

Next, we perform a calculation which is of direct engineering significance. We consider the boundary layer on the wall of a supersonic nozzle (see [18]). Transition in this boundary layer is caused by Görtler vortices. The flow accelerates to Mach 3.5 towards the exit of the nozzle. The distribution of the local edge Mach number, the pressure gradient parameter (β_h), and the Görtler number based upon momentum thickness θ are plotted in Fig. 5. The amplification factor ($N = \int^x \sigma(x)\, dx$) for Görtler vortices is computed for a fixed physical wavelength (local nondimensional wavenumber is also plotted in the figure) using the current asymptotic theory and the parallel theory used in the computations of [18]. For design purposes, the agreement between the two approaches is fairly good, but it should be noted that the parallel-theory calculations requires at least 30 times as much computer time when compared with the asymptotic calculations.

Appendix A

The non-zero elements of the coefficient matrices C, E, F (equation 3.11) are

$$c_{11} = -\mu_2 - \frac{\beta_0}{\bar{T}_0}\left[\bar{u}_2 - \frac{\bar{u}_1\bar{T}_1}{\bar{T}_0} + \bar{u}_0\left(\left[\frac{\bar{T}_1}{\bar{T}_0}\right]^2 - \frac{\bar{T}_2}{\bar{T}_0}\right)\right] + \frac{\alpha}{\bar{T}_0}\left[\frac{\bar{T}_1\bar{u}_1}{\bar{T}_0} - 2\bar{u}_2\right],$$

$$c_{12} = \frac{1}{\bar{T}_0}\left[-3\bar{u}_3 + 2\frac{\bar{u}_2\bar{T}_1}{\bar{T}_0} - \bar{u}_1\left(\left(\frac{\bar{T}_1}{\bar{T}_0}\right)^2 - \frac{\bar{T}_2}{\bar{T}_0}\right)\right],$$

$$c_{21} = \frac{g_0}{\bar{T}_0}\left[-\bar{u}_2 + \frac{\bar{u}_1\bar{T}_1}{\bar{T}_0} - \bar{u}_0\left(\left(\frac{\bar{T}_1}{\bar{T}_0}\right)^2 - \frac{\bar{T}_2}{\bar{T}_0}\right)\right]$$

$$\qquad - \alpha\left[u_1 + \frac{\beta_0}{\bar{T}_0}\left(\bar{u}_1 - \frac{\bar{T}_1\bar{u}_0}{\bar{T}_0}\right)\right] + \frac{\beta g_0\bar{u}_0}{\bar{T}_0^2}\left(\bar{u}_1 - \frac{\bar{u}_0\bar{T}_1}{\bar{T}_0}\right),$$

$$c_{22} = -\mu_2 - \frac{\beta_0}{\bar{T}_0}\left[\bar{u}_2 - \frac{\bar{u}_1\bar{T}_1}{\bar{T}_0} + \bar{u}_0\left\{\left(\frac{\bar{T}_1}{\bar{T}_0}\right)^2 - \frac{\bar{T}_2}{\bar{T}_0}\right\}\right],$$

$$c_{23} = \frac{-g_0}{2\bar{T}_0^2}\left[2\bar{u}_0\left\{u_2 - \frac{\bar{u}_1\bar{T}_1}{\bar{T}_0} + \bar{u}_0\left[\left(\frac{\bar{T}_1}{\bar{T}_0}\right)^2 - \frac{\bar{T}_2}{\bar{T}_0}\right]\right\} + \left(\bar{u}_1 - \frac{\bar{T}_1\bar{u}_0}{\bar{T}_0}\right)^2\right],$$

$$c_{31} = -\frac{\alpha}{\bar{T}_0}\left[2\bar{T}_2 - \frac{\bar{T}_1^2}{\bar{T}_0}\right] - \beta\left[\left(\bar{u}_1 - \frac{\bar{T}_1\bar{u}_0}{\bar{T}_0}\right)\frac{\beta_0}{\bar{T}_0} - \frac{\mu_1\bar{T}_1}{\Gamma}\right],$$

$$c_{32} = -\frac{1}{\bar{T}_0}\left[3\bar{T}_3 - \frac{3\bar{T}_1\bar{T}_2}{\bar{T}_0} + \frac{\bar{T}_1^3}{\bar{T}_0^2}\right],$$

$$c_{33} = -\left[\frac{\mu_2}{\Gamma} + \left\{\left[\bar{u}_2 - \frac{\bar{u}_1\bar{T}_1}{\bar{T}_0} + \bar{u}_0\left(\left(\frac{\bar{T}_1}{\bar{T}_0}\right)^2 - \frac{\bar{T}_2}{\bar{T}_0}\right)\right\}\frac{\beta_0}{\bar{T}_0}\right],$$

$$E_{11} = \bar{\mu}_0, \quad E_{22} = 2\bar{\mu}_0, \quad E_{33} = \frac{\bar{\mu}_0}{\Gamma},$$

$$F_{11} = -\frac{\bar{u}_0}{\bar{T}_0}, \quad F_{22} = -\frac{\bar{u}_0}{\bar{T}_0}, \quad F_{33} = -\frac{\bar{u}_0}{T_0}.$$

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